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New sum rules for Racah and Clebsch–Gordan coefficients

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Abstract. Making use of simple properties of the angular momentum algebra, new sum rules involving Legendre polynomials, Clebsch–Gordan and Racah coefficients are derived, and their connection with collision processes is illustrated. When the argument of the Legendre polynomials is set to unity, some of the obtained sum rules reduce to well known formulae. Finally, employing a recurrence relation for Racah coefficients, further sum rules are presented.

1. Introduction

In this paper we give and derive new sum rules involving Legendre polynomials, Clebsch–Gordan and Racah coefficients, using simple properties of the angular momentum algebra. To do so we consider the sum

$$S^{LL'}_{\ell^a_i\ell^a_f}(\ell_i\ell_f,\ell'_i\ell'_f;\theta)=\left[(2\ell_i+1)(2\ell'_i+1)(2\ell_f+1)(2\ell'_f+1)(2L+1)(2L'+1)\right]^{1/2}$$
$$\sum_{\mathcal{L}=0}^{\infty}(2\mathcal{L}+1)\begin{Bmatrix}\ell_i & \ell'_i & \mathcal{L}\\ L' & L & \ell^a_i\end{Bmatrix}\begin{Bmatrix}\ell_f & \ell'_f & \mathcal{L}\\ L' & L & \ell^a_f\end{Bmatrix}\begin{pmatrix}\ell_i & \ell'_i & \mathcal{L}\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\ell_f & \ell'_f & \mathcal{L}\\ 0 & 0 & 0\end{pmatrix}P_{\mathcal{L}}(\cos\theta) \tag{1}$$

where all ℓs are positive integers and $P_{\mathcal{L}}$ is a Legendre polynomial of degree $\mathcal{L}$. The multivariable angular function $S^{LL'}_{\ell^a_i\ell^a_f}(\ell_i\ell_f,\ell'_i\ell'_f;\theta)$ is found in the general formulation of the differential cross section for non-relativistic excitations by electron impact of atomic systems (see, for example, Blatt and Biedenharn 1952). In this case the above integers represent angular momenta. For partial wave analysis of collision problems and for these processes in particular, it would be useful to transform sum (1) so that a compact form for the differential cross section can be obtained (Ancarani 1992).

For excitation problems the meaning of the notation used in (1) is as follows. Initially the angular momenta of the target (superscript a) and of the colliding electron are respectively ℓ^a_i and ℓ_i; after the inelastic collision they are ℓ^a_f and ℓ_f. The total angular momentum L is obtained by coupling ℓ^a_i with ℓ_i and ℓ^a_f with ℓ_f; θ is the scattering angle. Note that $\mathcal{L}$, which is coupled to L and L', has no physical meaning. Both $\ell_i+\ell'_i+\mathcal{L}$ and $\ell_f+\ell'_f+\mathcal{L}$ must be even to prevent sum (1) vanishing. The geometrical representation of the coupling of angular momenta is shown in figure 1.

In section 2 we write sum (1) in a very different way. Section 3 deals with the special case $\ell^a_i=0$ and explicit results for $\ell^a_f=0,1$ and 2 are given. From some of these, further sum rules are derived (section 4) using a three-terms recurrence relation for Racah coefficients (Kachurik and Klimyk 1990). Throughout the paper, extensive use of the book by Edmonds (1957) on angular momentum will be made so that, for simplicity, this reference will be denoted by I in what follows.

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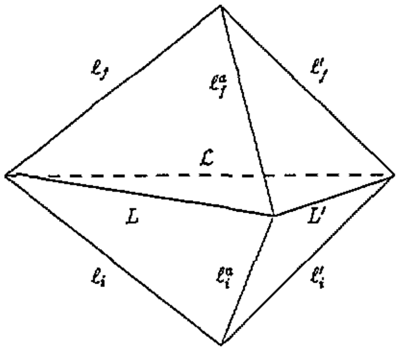

Figure 1. Coupling of angular momenta.

2. General case

Let us employ properties of the angular momentum algebra to write sum (1) in a different way. Making use of equation (6.2.8) of I, we write the coefficients of (1) as

$$\begin{Bmatrix} \ell'_i & \ell_i & \mathcal{L} \\ L & L' & \ell_i^a \end{Bmatrix} \begin{pmatrix} \ell_i & \ell'_i & \mathcal{L} \\ 0 & 0 & 0 \end{pmatrix} = \sum_{m_i=-\ell_i^a}^{\ell_i^a} (-)^{m_i+\ell_i^a+L+L'} \times \begin{pmatrix} \ell'_i & L' & \ell_i^a \\ 0 & m_i & -m_i \end{pmatrix} \begin{pmatrix} L & \ell_i & \ell_i^a \\ -m_i & 0 & m_i \end{pmatrix} \begin{pmatrix} L & L' & \mathcal{L} \\ m_i & -m_i & 0 \end{pmatrix} \tag{2a}$$

$$\begin{Bmatrix} \ell'_f & \ell_f & \mathcal{L} \\ L & L' & \ell_f^a \end{Bmatrix} \begin{pmatrix} \ell_f & \ell'_f & \mathcal{L} \\ 0 & 0 & 0 \end{pmatrix} = \sum_{m_f=-\ell_f^a}^{\ell_f^a} (-)^{m_f+\ell_f^a+L+L'} \times \begin{pmatrix} \ell'_f & L' & \ell_f^a \\ 0 & m_f & -m_f \end{pmatrix} \begin{pmatrix} L & \ell_f & \ell_f^a \\ -m_f & 0 & m_f \end{pmatrix} \begin{pmatrix} L & L' & \mathcal{L} \\ m_f & -m_f & 0 \end{pmatrix} \tag{2b}$$

and replace $P_{\mathcal{L}}(\cos\theta)$ by $d_{00}^{(\mathcal{L})}(\theta)$ ($d_{m'm}^{(j)}(\theta) = \mathcal{D}_{m'm}^{(j)}(0\theta 0)$ where $\mathcal{D}_{m'm}^{(j)}(\omega)$ are the matrix elements of finite rotations corresponding to Euler angles $\omega = (\alpha\theta\gamma)$—see equation (4.1.10) of I). Using relations (4.3.2) and (4.2.5) of I, the summation over $\mathcal{L}$ can now be carried out:

$$\sum_{\mathcal{L}=0}^{\infty} (2\mathcal{L}+1) \begin{pmatrix} L & L' & \mathcal{L} \\ m_i & -m_i & 0 \end{pmatrix} \begin{pmatrix} L & L' & \mathcal{L} \\ m_f & -m_f & 0 \end{pmatrix} P_{\mathcal{L}}(\cos\theta) = (-)^{m_f-m_i}\, d_{m_i m_f}^{(L)}(\theta)\, d_{m_i m_f}^{(L')}(\theta)\,. \tag{3}$$

Hence, with these results, sum (1) becomes

$$S_{\ell_i^a \ell_f^a}^{LL'}(\ell_i\ell_f, \ell'_i\ell'_f; \theta) = \left[(2\ell_i+1)(2\ell'_i+1)(2\ell_f+1)(2\ell'_f+1)(2L+1)(2L'+1)\right]^{1/2} \times (-)^{\ell_i^a+\ell_f^a} \sum_{m_i=-\ell_i^a}^{\ell_i^a} \begin{pmatrix} L & \ell_i & \ell_i^a \\ -m_i & 0 & m_i \end{pmatrix} \begin{pmatrix} L' & \ell'_i & \ell_i^a \\ -m_i & 0 & m_i \end{pmatrix} \times \sum_{m_f=-\ell_f^a}^{\ell_f^a} \begin{pmatrix} L & \ell_f & \ell_f^a \\ -m_f & 0 & m_f \end{pmatrix} \begin{pmatrix} L' & \ell'_f & \ell_f^a \\ -m_f & 0 & m_f \end{pmatrix} d_{m_i m_f}^{(L)}(\theta)\, d_{m_i m_f}^{(L')}(\theta)\,. \tag{4}$$

In this form, the symmetry in the angular momenta is more apparent than in expression (1). The integer $\mathcal{L}$ (which has no physical meaning) has been eliminated and replaced by the projections of ℓ_i^a and ℓ_f^a.

3. Special case : $\ell_i^a = 0$

Let us now consider the special case $\ell_i^a = 0$. From (1) it follows immediately that $\ell_i = L$, $\ell_i' = L'$ and

$$S_{0\ell_f^a}^{LL'}(L\ell_f, L'\ell_f'; \theta) = \left[(2\ell_f+1)(2\ell_f'+1)(2L+1)(2L'+1)\right]^{1/2} \times \sum_{\mathcal{L}=0}^{\infty}(2\mathcal{L}+1)\begin{Bmatrix} \ell_f & \ell_f' & \mathcal{L} \\ L' & L & \ell_f^a \end{Bmatrix}\begin{pmatrix} L & L' & \mathcal{L} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \ell_f & \ell_f' & \mathcal{L} \\ 0 & 0 & 0 \end{pmatrix} P_{\mathcal{L}}(\cos\theta) . \tag{5}$$

On the other hand, $m_i = 0$ and (4) reduces to

$$S_{0\ell_f^a}^{LL'}(L\ell_f, L'\ell_f'; \theta) = \left[(2\ell_f+1)(2\ell_f'+1)(2L+1)(2L'+1)\right]^{1/2}(-)^{\ell_f^a+L+L'} \times \sum_{m_f=-\ell_f^a}^{\ell_f^a}\begin{pmatrix} L & \ell_f & \ell_f^a \\ -m_f & 0 & m_f \end{pmatrix}\begin{pmatrix} L' & \ell_f' & \ell_f^a \\ -m_f & 0 & m_f \end{pmatrix} \times \left[\frac{(L-m_f)!\,(L'-m_f)!}{(L+m_f)!\,(L'+m_f)!}\right]^{1/2} P_L^{m_f}(\cos\theta)P_{L'}^{m_f}(\cos\theta) \tag{6}$$

where $P_L^m(\cos\theta)$ $(m = 0, \pm1, \ldots, \pm L)$ are the associated Legendre polynomials of the first kind of degree L and order m defined by equation (2.5.17) of I. As regards the physical application referred to in the introduction, expression (5) can be found in part of formula (II.B.47) of the paper by Alder *et al* (1956). Form (6) of sum (1) has proved to be very useful in deriving a compact formulation for the differential cross section corresponding to excitations by electron impact of atomic systems in which the initial ($\ell_i^a = 0$) and/or final ($\ell_f^a = 0$) state is an s-state (see Ancarani 1992). Note that, from the algebraic point of view, the two cases are the same.

When the angle θ is set to zero (this corresponds to the forward scattering direction) sum (6) vanishes unless $m_f = 0$. Moreover $P_\ell(1) = 1\ \forall\ell$, and relations (5) and (6) become

$$S_{0\ell_f^a}^{LL'}(L\ell_f, L'\ell_f'; 0) = \left[(2\ell_f+1)(2\ell_f'+1)(2L+1)(2L'+1)\right]^{1/2} \times \sum_{\mathcal{L}=0}^{\infty}(2\mathcal{L}+1)\begin{Bmatrix} \ell_f & \ell_f' & \mathcal{L} \\ L' & L & \ell_f^a \end{Bmatrix}\begin{pmatrix} L & L' & \mathcal{L} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \ell_f & \ell_f' & \mathcal{L} \\ 0 & 0 & 0 \end{pmatrix} \tag{7a}$$

$$= \left[(2\ell_f+1)(2\ell_f'+1)(2L+1)(2L'+1)\right]^{1/2}(-)^{\ell_f^a+L+L'} \times \begin{pmatrix} L & \ell_f & \ell_f^a \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} L' & \ell_f' & \ell_f^a \\ 0 & 0 & 0 \end{pmatrix} \tag{7b}$$

the second equality being a well known result (see equation (6.2.6) of I). Note that (7*a*) can also be written in the notation of I as

$$(-)^{L+L'+\ell_f^a}\sum_{\mathcal{L}=0}^{\infty}(\ell_f L \ell_f^a|P_{\mathcal{L}}(\cos\theta)|\ell_f' L' \ell_f^a) .$$

We now focus on three specific values of ℓ_f^a, namely 0, 1 and 2. For the collision problem considered they correspond respectively to final s-, p- and d-states of the atomic system i.e. monopole, dipole and quadrupole transitions. For simplicity, the argument of the Legendre polynomials will be omitted in what follows.

3.1. $\ell_{\rm f}^{\rm a} = 0$

In this very simple case we have $m_{\rm f} = 0, \ell_{\rm f} = L, \ell_{\rm f}' = L'$ and

$$S_{00}^{LL'}(LL, L'L'; \theta) = \left[(2L+1)(2L'+1)\right]^{1/2} P_L P_{L'} \tag{8a}$$

$$S_{00}^{LL'}(LL, L'L'; 0) = \left[(2L+1)(2L'+1)\right]^{1/2}. \tag{8b}$$

3.2. $\ell_{\rm f}^{\rm a} = 1$

There are three possible values for $m_{\rm f}$ $(-1, 0, 1)$; after some algebra, (6) becomes

$$\begin{aligned} S_{01}^{LL'}(L\ell_{\rm f}, L'\ell_{\rm f}'; \theta) &= \left[(2\ell_{\rm f}+1)(2\ell_{\rm f}'+1)(2L+1)(2L'+1)\right]^{1/2}(-)^{1+L+L'} \\ &\quad \times \left[\begin{pmatrix} L & \ell_{\rm f} & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L' & \ell_{\rm f}' & 1 \\ 0 & 0 & 0 \end{pmatrix} P_L P_{L'} \right. \\ &\quad \left. + \begin{pmatrix} L & \ell_{\rm f} & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} L' & \ell_{\rm f}' & 1 \\ -1 & 0 & 1 \end{pmatrix} \frac{2P_L^1 P_{L'}^1}{[L(L+1)L'(L'+1)]^{1/2}} \right]. \end{aligned} \tag{9}$$

From figure 1 it is straightforward to deduce the restricting triangular inequalities

$$|L-1| \leqslant \ell_{\rm f} \leqslant |L+1| \qquad |L'-1| \leqslant \ell_{\rm f}' \leqslant |L'+1| \tag{10}$$

and, taking into account that $L + L' + \ell_{\rm f} + \ell_{\rm f}'$ is even, there are five possible pairs $(\ell_{\rm f}, \ell_{\rm f}')$ for which the results are

$$S_{01}^{LL'}(LL-1, L'L'-1; \theta) = -[LL']^{1/2} P_L P_{L'} - [LL']^{-\frac{1}{2}} P_L^1 P_{L'}^1 \tag{11a}$$

$$S_{01}^{LL'}(LL-1, L'L'+1; \theta) = [L(L'+1)]^{1/2} P_L P_{L'} - [L(L'+1)]^{-\frac{1}{2}} P_L^1 P_{L'}^1 \tag{11b}$$

$$S_{01}^{LL'}(LL+1, L'L'-1; \theta) = [(L+1)L']^{1/2} P_L P_{L'} - [(L+1)L']^{-\frac{1}{2}} P_L^1 P_{L'}^1 \tag{11c}$$

$$S_{01}^{LL'}(LL+1, L'L'+1; \theta) = -[(L+1)(L'+1)]^{1/2} P_L P_{L'} - [(L+1)(L'+1)]^{-\frac{1}{2}} P_L^1 P_{L'}^1 \tag{11d}$$

$$S_{01}^{LL'}(LL, L'L'; \theta) = -\left[\frac{(2L+1)(2L'+1)}{L(L+1)L'(L'+1)}\right]^{\frac{1}{2}} P_L^1 P_{L'}^1 \tag{11e}$$

and

$$S_{01}^{LL'}(LL-1, L'L'-1; 0) = -[LL']^{1/2} \tag{12a}$$

$$S_{01}^{LL'}(LL-1, L'L'+1; 0) = [L(L'+1)]^{1/2} \tag{12b}$$

$$S_{01}^{LL'}(LL+1, L'L'-1; 0) = [(L+1)L']^{1/2} \tag{12c}$$

$$S_{01}^{LL'}(LL+1, L'L'+1; 0) = -[(L+1)(L'+1)]^{1/2} \tag{12d}$$

$$S_{01}^{LL'}(LL, L'L'; 0) = 0. \tag{12e}$$

3.3. $\ell_f^a = 2$

In this case m_f takes the values $-2, -1, 0, 1, 2$ and relation (6) becomes

$$S_{02}^{LL'}(L\ell_f, L'\ell_f'; \theta) = \left[(2\ell_f+1)(2\ell_f'+1)(2L+1)(2L'+1)\right]^{1/2}(-)^{L+L'} \times \left[\begin{pmatrix} L & \ell_f & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} L' & \ell_f' & 2 \\ 0 & 0 & 0 \end{pmatrix} P_L P_{L'} + \begin{pmatrix} L & \ell_f & 2 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} L' & \ell_f' & 2 \\ -1 & 0 & 1 \end{pmatrix} \frac{2\,P_L^1 P_{L'}^1}{[L(L+1)\,L'(L'+1)]^{1/2}} + \begin{pmatrix} L & \ell_f & 2 \\ -2 & 0 & 2 \end{pmatrix}\begin{pmatrix} L' & \ell_f' & 2 \\ -2 & 0 & 2 \end{pmatrix} \times \frac{2\,P_L^2 P_{L'}^2}{[(L-1)L(L+1)(L+2)\,(L'-1)L'(L'+1)(L'+2)]^{1/2}}\right]. \tag{13}$$

The 13 pairs (ℓ_f, ℓ_f') giving non-vanishing values of (13) satisfy the following conditions:

$$|L-2| \leqslant \ell_f \leqslant |L+2| \qquad |L'-2| \leqslant \ell_f' \leqslant |L'+2| \qquad L+L'+\ell_f+\ell_f' = \text{even} \tag{14}$$

and are

$$(|L-2|, |L'-2|) \qquad (|L-2|, L') \qquad (|L-2|, L'+2) \tag{15a}$$
$$(|L-1|, |L'-1|) \qquad (|L-1|, L'+1) \tag{15b}$$
$$(L, |L'-2|) \qquad (L, L') \qquad (L, L'+2) \tag{15c}$$
$$(L+1, |L'-1|) \qquad (L+1, L'+1) \tag{15d}$$
$$(L+2, |L'-2|) \qquad (L+2, L') \qquad (L+2, L'+2)\,. \tag{15e}$$

The explicit values of $S_{02}^{LL'}(L\ell_f, L'\ell_f'; \theta)$ for these pairs can be easily calculated using table 2 of I. For example, considering the pair (L, L') one obtains

$$S_{02}^{LL'}(LL, L'L'; \theta) = \left[\frac{(2L+1)(2L'+1)}{(2L-1)(2L+3)(2L'-1)(2L'+3)}\right]^{1/2} \times \left[[L(L+1)L'(L'+1)]^{1/2} P_L P_{L'} + \frac{3\,(P_L^1 P_{L'}^1 + P_L^2 P_{L'}^2)}{[L(L+1)L'(L'+1)]^{1/2}}\right] \tag{16a}$$

and

$$S_{02}^{LL'}(LL, L'L'; 0) = \left[\frac{L(L+1)L'(L'+1)(2L+1)(2L'+1)}{(2L-1)(2L+3)(2L'-1)(2L'+3)}\right]^{1/2}. \tag{16b}$$

Relation (6) can be examined for larger values of ℓ_f^a but the formulae become longer and are of less interest for the application referred to in the introduction. Note that two more relations have to be satisfied if parity is conserved in the collision process

$$(-)^{\ell_i^a+\ell_i} = (-)^{\ell_f^a+\ell_f} \qquad (-)^{\ell_i^a+\ell_i'} = (-)^{\ell_f^a+\ell_f'}. \tag{17}$$

As a consequence, in the case $\ell_i^a = 0$, some pairs (ℓ_f, ℓ_f') are not physically allowed, namely (L, L') for $\ell_f^a = 1$, and the pairs (15*b*) and (15*d*) for $\ell_f^a = 2$. When neither ℓ_f^a nor ℓ_i^a is zero the rotation matrix elements do not reduce to simple Legendre polynomials but involve Jacobi polynomials $P_\ell^{(m,m')}(\cos\theta)$ (see equation (4.1.23) of I). We will not consider here any of these cases.

4. Further sum rules

Further sum rules can be derived from the above formulae. When $m_\mathrm{i} = m_\mathrm{f} = 0$, relation (3) reduces to

$$\sum_{\mathcal{L}=0}^{\infty}(2\mathcal{L}+1)\begin{pmatrix} L & L' & \mathcal{L} \\ 0 & 0 & 0 \end{pmatrix}^2 P_{\mathcal{L}} = P_L P_{L'} \tag{18a}$$

and for $\theta = 0$

$$\sum_{\mathcal{L}=0}^{\infty}(2\mathcal{L}+1)\begin{pmatrix} L & L' & \mathcal{L} \\ 0 & 0 & 0 \end{pmatrix}^2 = 1\,. \tag{18b}$$

In what follows we assume that $L + L' + \mathcal{L}$ is even. Making use of

$$\begin{Bmatrix} L & L' & \mathcal{L} \\ L' & L & 1 \end{Bmatrix} = \frac{\mathcal{L}(\mathcal{L}+1) - [L(L+1) + L'(L'+1)]}{2[\,L(2L+1)(L+1)\,L'(2L'+1)(L'+1)]^{1/2}} \tag{19}$$

and of (11e) and (18a), it is easy to deduce the following sum rules:

$$\sum_{\mathcal{L}=0}^{\infty}(2\mathcal{L}+1)\,\mathcal{L}(\mathcal{L}+1)\begin{pmatrix} L & L' & \mathcal{L} \\ 0 & 0 & 0 \end{pmatrix}^2 P_{\mathcal{L}} = [L(L+1) + L'(L'+1)]\,P_L P_{L'} - 2\,P_L^1 P_{L'}^1 \tag{20a}$$

$$\sum_{\mathcal{L}=0}^{\infty}(2\mathcal{L}+1)\,\mathcal{L}(\mathcal{L}+1)\begin{pmatrix} L & L' & \mathcal{L} \\ 0 & 0 & 0 \end{pmatrix}^2 = L(L+1) + L'(L'+1)\,. \tag{20b}$$

Similarly, using (16a), (18a), (20a) and

$$\begin{Bmatrix} L & L' & \mathcal{L} \\ L' & L & 2 \end{Bmatrix} = [3X(X-1) - 4L(L+1)L'(L'+1)]/\{2[(2L-1)L(2L+1)(L+1) \times (2L+3)(2L'-1)L'(2L'+1)(L'+1)(2L'+3)]^{1/2}\} \tag{21}$$

where $X = L(L+1) + L'(L'+1) - \mathcal{L}(\mathcal{L}+1)$, one obtains

$$\begin{aligned}\sum_{\mathcal{L}=0}^{\infty}(2\mathcal{L}+1)\,\mathcal{L}^2(\mathcal{L}+1)^2\begin{pmatrix} L & L' & \mathcal{L} \\ 0 & 0 & 0 \end{pmatrix}^2 P_{\mathcal{L}} &= \left\{[L(L+1) + L'(L'+1)]^2 + 2L(L+1)L'(L'+1)\right\} P_L P_{L'} \\ &\quad + 4[1 - L(L+1) - L'(L'+1)]P_L^1 P_{L'}^1 + 2\,P_L^2 P_{L'}^2\end{aligned} \tag{22a}$$

and

$$\sum_{\mathcal{L}=0}^{\infty}(2\mathcal{L}+1)\,\mathcal{L}^2(\mathcal{L}+1)^2\begin{pmatrix} L & L' & \mathcal{L} \\ 0 & 0 & 0 \end{pmatrix}^2 = [L(L+1) + L'(L'+1)]^2 + 2L(L+1)L'(L'+1)\,. \tag{22b}$$

Following this procedure, further sum rules of the kind (22a) and (22b) (with $\mathcal{L}^n(\mathcal{L}+1)^n$, $n \geqslant 3$) can be obtained considering larger values of $\ell_\mathrm{f}^\mathrm{a}$.

The $\theta = 0$ formulae (18*b*), (20*b*) and (22*b*) can be easily generalized using a result of the paper by Kachurik and Klimyk (1990). Indeed, their recurrence formula (16) relates the three Racah coefficients

$$\begin{Bmatrix} a & b & c \\ d+1 & e & f \end{Bmatrix} \qquad \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} \qquad \begin{Bmatrix} a & b & c \\ d-1 & e & f \end{Bmatrix}$$

allowing us to obtain further sum rules such as

$$\sum_{\mathcal{L}=0}^{\infty}(2\mathcal{L}+1)\,\mathcal{L}^3(\mathcal{L}+1)^3 \begin{pmatrix} L & L' & \mathcal{L} \\ 0 & 0 & 0 \end{pmatrix}^2 = [L(L+1)+L'(L'+1)]^3 + 2L(L+1)L'(L'+1)\big[3[L(L+1)+L'(L'+1)]-2\big] \tag{23}$$

and

$$\sum_{\mathcal{L}=0}^{\infty}(2\mathcal{L}+1)\,\mathcal{L}^4(\mathcal{L}+1)^4 \begin{pmatrix} L & L' & \mathcal{L} \\ 0 & 0 & 0 \end{pmatrix}^2 = [L(L+1)+L'(L'+1)]^4 + 2L(L+1)L'(L'+1)\big\{8+3L(L+1)L'(L'+1) + 2[L(L+1)+L'(L'+1)]\big[3[L(L+1)+L'(L'+1)]-5\big]\big\}\,. \tag{24}$$

Finally, using some of the previous results, we can write the following formula:

$$\sum_{\mathcal{L}=0}^{\infty}(2\mathcal{L}+1) \begin{Bmatrix} L & L' & \mathcal{L} \\ L' & L & 1 \end{Bmatrix}^2 \begin{pmatrix} L & L' & \mathcal{L} \\ 0 & 0 & 0 \end{pmatrix}^2 = \frac{1}{2(2L+1)(2L'+1)} \tag{25}$$

for $L, L' \geqslant 1$ (when $L = 0$ or $L' = 0$ the sum is zero).

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